Kuwait University | Math 102 | July 27, 2006
Dept of Math & Comp Sci | Final Examination | Duration: 2 hours

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The use of calculator is not allowed. Turn off your mobile phone or pager

1. Let $f(x) = \tan^{-1} x + \tan^{-1}(\ln x) - \frac{\pi}{4}$, $x > 0$.

   (a) Show that f is one-to-one over the interval $(0, \infty)$. **(1 point)**

   (b) Find the domain of f^{-1}. **(1 point)**

   (c) Show that $P(0, 1)$ is on the graph of f^{-1} and find the slope of the tangent line at P. **(2 points)**

2. Evaluate $\lim_{x\to 0^+} (\sin x)^{\frac{1}{\ln x}}$, if it exists. **(3 points)**

3. Evaluate the following integrals. **(3 points each)**

   (a) $\int \frac{\sin 2x}{1+\sin^2 x}\,dx$ (b) $\int \frac{\sqrt{x^2-4}}{x^2}\,dx$ (c) $\int 2x^3 (\sec x^2)^2\,dx$ (d) $\int \frac{\cos^3 x}{\sqrt{1+\sin x}}\,dx$

4. Determine if the improper integral $\int_1^4 \frac{1}{\sqrt{8+2x-x^2}}\,dx$ is convergent or divergent, and find its value if it is convergent. **(3 points)**

5. Find the length of the cardioid $r = 1 + \sin\theta$. **(4 points)**

6. Find the equation, in rectangular coordinates of the tangent line to the graph of the polar equation $r = 2 + \sin\theta$ at the point that corresponds to $\theta = 0$ **(4 points)**

7. Sketch the graphs of the polar equations $r = \cos\theta$ and $r = 1 - \cos\theta$, and find the area that lies inside both graphs. **(4 points)**

8. Consider the curve C given by the parametric equations

$$x = e^{\frac{t}{2}} \cos t \qquad y = e^{\frac{t}{2}} \sin t, \qquad 0 \le t \le 2\pi$$

   (a) Show that **(2 points)**

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{5}{4}e^t$$

   (b) The curve C is revolved about the x-axis. Find the area of the resulting surface. **(4 points)**

(a) $f'(x) = \frac{1}{1+x^2} + \frac{1}{x(1+\ln^2 x)} > 0$ for all $x > 0$. So f is increasing on $(0,\infty)$ and therefore is (1-1) there.

(b) $D_{f^{-1}} = R_f = (\lim_{x\to 0^+} f(x), \lim_{x\to\infty} f(x)) = (-\frac{3\pi}{4}, \frac{3\pi}{4})$

(c) $f(1) = 0$, so $P(0,1)$ is on the graph of f^{-1}. The slope $= \frac{df^{-1}(0)}{dx} = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$.

2. $y = (\sin x)^{\frac{1}{\ln x}}$, $\ln y = \frac{\ln\sin x}{\ln x}$. $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln \sin x}{\ln x} = \lim_{x\to 0^+} \frac{x}{\sin x}\cos x = 1$. $\lim_{x\to 0^+} (\sin x)^{\frac{1}{\ln x}} = e$.

3.

(a) $\int \frac{\sin 2x}{1+\sin^2 x}dx = \int \frac{2\sin x\cos x}{1+\sin^2 x}dx \overset{u=\sin x}{=} \int \frac{2u}{1+u^2}du = \ln(1+\sin^2 x) + C$

(b) $\int \frac{\sqrt{x^2-4}}{x^2}dx \overset{x=2\sec\theta}{=} \int \frac{2\tan\theta}{4\sec^2\theta} 2\sec\theta\tan\theta d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int (\sec\theta - \cos)\theta d\theta$

$= \ln|\sec\theta + \tan\theta| - \sin\theta + C = \ln\left|x + \sqrt{x^2-4}\right| - \frac{\sqrt{x^2-4}}{x} + C$

(c) $\int 2x^3\sec^2 x^2 dx = \int x^2\left(2x\sec^2 x^2\right)dx = \int x^2\left(\tan x^2\right)' dx = x^2\left(\tan x^2\right) - \int 2x\tan x^2 dx$

$= x^2\left(\tan x^2\right) + \ln\left|\cos x^2\right| + C$

(d) $\int \frac{\cos^3 x}{\sqrt{1+\sin x}}dx = \int\left(1-\sin^2 x\right)(1+\sin x)^{-\frac{1}{2}}\cos x dx = \int \frac{1-u^2}{\sqrt{1+u}}du$

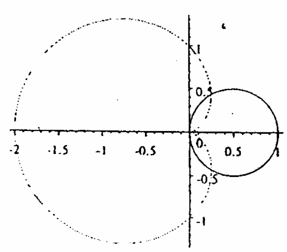
$= -\frac{2}{5}(1+u)^{\frac{5}{2}} + \frac{4}{3}(1+u)^{\frac{3}{2}} = -\frac{2}{5}(1+\sin x)^{\frac{5}{2}} + \frac{4}{3}(1+\sin x)^{\frac{3}{2}}$

4. $\int_1^4 \frac{dx}{\sqrt{8+2x-x^2}} = \lim_{t\to 4^-}\int_1^t \frac{dx}{\sqrt{9-(x-1)^2}} = \lim_{t\to 4^-} \sin^{-1}\left(\frac{x-1}{3}\right)\Big|_1^t = \sin^{-1} 1 = \frac{\pi}{2}$

5. $2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1+\sin\theta)}d\theta = 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin\theta}}d\theta = -4\sqrt{2}\sqrt{1-\sin\theta}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 8.$

6. $\frac{dr}{d\theta} = \cos\theta$. The slope $= \left.\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\right|_{\theta=0} = \left.\frac{\cos\theta\sin\theta + (2+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (2+\sin\theta)\sin\theta}\right|_{\theta=0} = 2.$

The point of tangency is $P(2,0)$. The equation is $y = 2(x-2)$.



7.

Area $= \int_0^{\frac{\pi}{3}}(1-\cos\theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta = \int_0^{\frac{\pi}{3}}(1-2\cos\theta)d\theta + \int_0^{\frac{\pi}{2}}\cos^2\theta = \frac{7}{12}\pi - \sqrt{3}$

8. (a) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^t\left(\frac{1}{2}\cos t - \sin t\right)^2 + e^t\left(\frac{1}{2}\sin t + \cot t\right)^2 = e^t\left(\frac{1}{4}+1\right) = \frac{5}{4}e^t$

(b) Area $= \int_0^{2\pi} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \frac{\sqrt{5}}{2}\int_0^{2\pi} e^t(\sin t)dt = \frac{\sqrt{5}}{4}\left(1 - e^{2\pi}\right)$ by using integration by parts repeatedly .